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Motion on a given surface: Monoparametric families of orbits sufficient for separability

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Abstract

In the light of the inverse problem of dynamics, we study the motion of a material point on an arbitrary two-dimensional surface, submersed in $\mathbb{E}^3$. Under the assumption that a monoparametric family of geodesics and their orthogonal trajectories form an isothermic coordinate system, we prove that, if the family of geodesics is a family of orbits of the material point, compatible with the potential, then the system is integrable with an integral linear in the velocities, while, compatibility of the potential with the orthogonal trajectories guarantees integrability with a quadratic integral of motion. In both cases, we determine the form of the potential modulo one or two arbitrary functions respectively and the corresponding form of the integral, while, for the case of the orthogonal trajectories, we determine the allowed regions of motion on the surface and their stability.

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1. Introduction

The inverse problem of dynamics in a broad sense consists of the determination of forces, parameters and constraints which are required for the realization of the motion of a mechanical system with some properties given in advance [8]. Szebehely [16] published a partial differential

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equation for the potential function $V = V(x, y)$ which produces a monoparametric family of planar orbits $f(x, y) = c$ and the energy E of them is given in advance as a function of the constant c namely $E = E(c)$. Mertens [14] studied a family of curves $f(u, v) = c$ on a surface S in 3-D space using Szebehely's method and obtained a linear partial differential equation in the potential function $V(u, v)$. Furthermore, Bozis and Mertens [3] derived a second order partial differential equation of hyperbolic type for the potential V in which all the coefficients are known functions of the coordinates u, v and gave some examples. Borghero [2] determined the expressions for the covariant components Q_1, Q_2 of forces acting on a test particle which describe orbits on a given surface, using the procedure of Dainelli [18]. Bozis and Borghero [5] introduced the notion of the family boundary curves (FBC) for that version of the inverse problem of dynamics which combines the potential $V(u, v)$ with a monoparametric family of regular orbits $f(u, v) = c$ on the configuration manifold (M_2, g) of a conservative holonomic system with $n = 2$ degrees of freedom. Several examples were given there. Kotoulas [11] studied solvable cases of the PDE in $V(u, v)$ given by Bozis and Mertens [3]. Moreover, Kotoulas [12] determined the generalized force field which gives rise to a *two*-parametric family of orbits on a given surface. A review on the basic facts of the inverse problem in dynamics was made by Bozis [4] and recently by Anisiu [1].

Besides providing a relation between permissible families of orbits and the potential, the inverse problem may supply additional information on the dynamics, as for example decisions on the integrability of the system through the existence of certain orbits, without knowing the potential. Ichtiaroglou and Meletlidou [10] have shown that, in the case of planar motion, the presence of a monoparametric family of conic sections or a family of confocal parabolas guarantees the integrability of the potential, with an integral of motion quadratic in the velocities. As special cases, they obtained that the presence of a family of straight lines, intersecting at a certain point, results to a central potential and a linear integral of motion (i.e. the angular momentum), while a family of concentric circles guarantees that the potential is separable in polar coordinates and thus possesses an integral, quadratic in the velocities. On the other hand, Voyatzi and Ichtiaroglou [17] studied motion on the two-dimensional sphere and showed that the permissibility of a family of meridians or a family of parallels guarantees integrability, with linear or quadratic integrals respectively.

Intrigued by these results, we study the motion of a material point on an arbitrary two-dimensional surface, submersed in $\mathbb{E}^3$. We select a coordinate system such that one family of coordinate lines comprises of geodesics and the other family of their orthogonal trajectories. Moreover we assume that this coordinate system is isothermic. We prove that, if the family of geodesics is a family of orbits of the material point, compatible with the potential, then it is integrable with an integral linear in the velocities, while compatibility of the potential with the orthogonal trajectories guarantees integrability with a quadratic integral of motion. In both cases, we determine the form of the potential modulo one or two arbitrary functions respectively and the corresponding form of the integral, while for the case of the orthogonal trajectories, we determine the allowed regions of motion on the surface and their stability.

In Section 2 we present the general setting of the inverse problem of dynamics related with the motion of a test particle on a given surface, using the selected coordinates. In Section 3 we study the case of integrability with an integral linear in the velocities, while in Section 4 we study the case of a quadratic integral. Several specific examples are presented in Section 5 and some concluding remarks in Section 6.

2. Motion on a surface

In Euclidean space $\mathbb{E}^3$ with an orthonormal Cartesian system of reference $Oxyz$ we assign a smooth surface S by the parametrization

$$\{x = x(u, v), y = y(u, v), z = z(u, v)\} \tag{1}$$

where u, v are curvilinear coordinates on S. On this surface we also consider a monoparametric family of regular curves given in the solved form

$$f(u, v) = c \tag{2}$$

where $c =$ const. is a parameter which varies along the family (2).

For the given family of orbits we define $\gamma = f_v/f_u$, where the subscripts denote partial differentiation with respect to the corresponding variable. The *slope function* γ represents the family (2) in the sense that if the family (2) is given, then γ is determined uniquely. On the other hand, if γ is given, we can obtain a unique family (2). The inverse problem of dynamics consists in finding potentials V which give rise to this family of orbits (2) on a given surface (1).

The line-element on the surface S in this system of parameters is given by

$$\mathrm{d}s^2 = g_{11}\mathrm{d}u^2 + 2g_{12}\mathrm{d}u\mathrm{d}v + g_{22}\mathrm{d}v^2 \tag{3}$$

where g_{11}, g_{12}, g_{22} are known functions of u, v.

Now, we consider a particle of unit mass which describes any member of the given family (2). Here we have to clear out that trajectories are bound to a given surface by constraints. The kinetic energy of the test particle is given by

$$T = \frac{1}{2}(g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2) \tag{4}$$

where the dot denotes differentiation with respect to time. The equations of motion of the test particle are

$$\begin{aligned} g_{11}\ddot{u} + g_{12}\ddot{v} + \frac{1}{2}(g_{11})_u(\dot{u})^2 + (g_{11})_v\dot{u}\dot{v} + [(g_{12})_v - \frac{1}{2}(g_{22})_u](\dot{v})^2 = -V_u, \\ g_{12}\ddot{u} + g_{22}\ddot{v} + \frac{1}{2}(g_{22})_v(\dot{v})^2 + (g_{22})_u\dot{u}\dot{v} + [(g_{12})_u - \frac{1}{2}(g_{11})_v](\dot{u})^2 = -V_v. \end{aligned} \tag{5}$$

Mertens[14] provided a linear, first order partial differential equation for the potential function $V = V(u, v)$ for any preassigned dependence $E = E(f)$, of the total energy E of the given family $f = f(u, v)$. This equation reads

$$(g_{22}f_u - g_{12}f_v)V_u + (g_{11}f_v - g_{12}f_u)V_v = 2W(E - V) \tag{6}$$

where W is given in the Appendix.

Moreover, Bozis and Mertens [3] derived a linear, second order partial differential equation in $V = V(u, v)$ which is independent of the total energy E and gives all the potential functions generating family (2) on the given surface (1). The total energy E must be constant along each orbit, so $E = E(f)$. So, we have $E_v = E_f f_v$ and $E_u = E_f f_u$. Hence, it is: $E_v = \gamma E_u$. Assuming that $W \neq 0$, Bozis and Mertens [3] obtained the following equation

$$k_1 V_{uu} + k_2 V_{uv} - \beta V_{vv} + k_3 V_u + k_4 V_v = 0 \tag{7}$$

where

$$k_1 = \alpha\gamma, \quad k_2 = \beta\gamma - \alpha, \quad k_3 = \gamma + \gamma\alpha_u - \alpha_v, \quad k_4 = \gamma\beta_u - \beta_v - 1. \tag{8}$$

The explicit form of the coefficients α, β, γ is also given in the Appendix.

In the present study, we shall consider the monoparametric family of geodesics $f(u, v) = v = c$ which induces a regular foliation on S. Then the members of the family $v = c$ can be used as coordinate lines. In this case we obtain from the equations of geodesics (see e.g. [7, p. 206])

$$\Gamma_{11}^2 = 0, \tag{9}$$

which means that $(g_{11})_v = 0$. Hence, we have

$$g_{11} = g_{11}(u). \tag{10}$$

We complete the set of coordinates on S selecting the family of curves $u = c$, orthogonal to $v = c$. So, we have

$$g_{12} = 0 \tag{11}$$

and the metric in (3) is written as

$$\mathrm{d}s^2 = g_{11}(u)\mathrm{d}u^2 + g_{22}(u, v)\mathrm{d}v^2, \tag{12}$$

while the equations of motion (5) take the simpler form

$$\begin{aligned} \ddot{u} &= \frac{1}{g_{11}}\left[-\frac{1}{2}(g_{11})_u\dot{u}^2 + \frac{1}{2}(g_{22})_u\dot{v}^2 - V_u\right], \\ \ddot{v} &= \frac{1}{g_{22}}\left[-\frac{1}{2}(g_{22})_v\dot{v}^2 - (g_{22})_u\dot{u}\dot{v} - V_v\right]. \end{aligned} \tag{13}$$

3. Existence of integrals of motion linear in the velocities

We shall seek for integrals of first degree in velocity components, i.e. integrals of the form

$$\Phi(u, v, \dot{u}, \dot{v}) = A(u, v)\dot{u} + B(u, v)\dot{v}. \tag{14}$$

Since the Lagrangian of the system has definite parity with respect to time inversion, the integral cannot possess terms of zeroth degree in the velocities. Otherwise these terms should be another integral of motion by themselves, see e.g. [9].

The total derivative of Φ with respect to time must be identically equal to zero, i.e.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A\ddot{u} + B\ddot{v} + \frac{\partial A}{\partial u}\dot{u}^2 + \frac{\partial B}{\partial v}\dot{v}^2 + \left(\frac{\partial A}{\partial v} + \frac{\partial B}{\partial u}\right)\dot{u}\dot{v} \equiv 0. \tag{15}$$

By replacing the expressions of $\ddot{u}$, $\ddot{v}$ from (13) into (15) and we obtain:

$$C_{11}\dot{u}^2 + C_{12}\dot{u}\dot{v} + C_{22}\dot{v}^2 + C_{10}V_u + C_{01}V_v \equiv 0 \tag{16}$$

where

$$
\begin{aligned}
C_{11} &= A_u - \frac{1}{2} A \frac{\partial(\log g_{11})}{\partial u}, \\
C_{12} &= A_v + B_u - B \frac{\partial(\log g_{22})}{\partial u}, \\
C_{22} &= B_v + \frac{1}{2} A \frac{(g_{22})_u}{g_{11}} - \frac{1}{2} B \frac{\partial(\log g_{22})}{\partial v}, \\
C_{10} &= -\frac{A}{g_{11}}, \\
C_{01} &= -\frac{B}{g_{22}}.
\end{aligned}
\tag{17}
$$

Since (16) holds by identity, we have

$$C_{11} = C_{12} = C_{22} = 0, \quad C_{10} V_u + C_{01} V_v = 0. \tag{18}$$

The last of Eqs. (18) is written as

$$\frac{A}{g_{11}} V_u + \frac{B}{g_{22}} V_v = 0. \tag{19}$$

At this point we assume that the monoparametric family of geodesics $f(u, v) = v = c$ are orbits compatible with the potential V. Then the expression of W in (6) is equal to zero [13]. Moreover, by taking into account (10), (11) and the fact that $f_u = 0$, the PDE (6) reads

$$g_{11} V_v = 0. \tag{20}$$

Comparing Eqs. (19) and (20), we get

$$A = 0 \tag{21}$$

and consequently $C_{11} = 0$ in (17). From (20) we ascertain that $V = V(u)$. We shall determine now the unknown function $B(u, v)$. From the requirement that $C_{12} = 0$ and $C_{22} = 0$, we obtain respectively

$$B(u, v) = b(v) g_{22}, \quad B(u, v) = a(u) \sqrt{g_{22}} \tag{22}$$

where $a(u)$, $b(v)$ are arbitrary C^2-functions. Thus, the component of the metric tensor g_{22} is equal to

$$g_{22} = \frac{a^2(u)}{b^2(v)} \tag{23}$$

i.e. the coordinates must form an *isothermic* system (see e.g. [7, p. 95]). Thus, the unknown function $B(u, v)$ is

$$B(u, v) = \frac{a^2(u)}{b(v)} \tag{24}$$

and the integral of motion becomes

$$\Phi(u, v, \dot{u}, \dot{v}) = \frac{a^2(u)}{b(v)} \dot{v}. \tag{25}$$

We shall prove now that the potential $V = V(u)$ is also compatible with the monoparametric family $f(u, v) = u = c$. Indeed, the PDE (7) reads

$$\alpha_v V_u = 0. \tag{26}$$

Since $V_u \neq 0$, we have: $\alpha_v = 0$ or, equivalently,

$$\frac{\partial^2(\log g_{22})}{\partial u \partial v} = 0. \tag{27}$$

Relation (27) is satisfied, since we have assumed that g_{22} is of the form (23). So, the potential $V = V(u)$, compatible with the family of geodesics $v = c$, is also compatible with the monoparametric family of their orthogonal trajectories $f(u, v) = u = c$. In the following we shall study the allowed region on S where a material point can trace these orbits and also their linear stability. Note that if these orbits are bounded, then they are periodic by necessity.

We consider the Lagrangian of the motion

$$\mathcal{L} = \frac{1}{2}(g_{11}\dot{u}^2 + g_{22}\dot{v}^2) - V(u). \tag{28}$$

The generalized momenta are

$$p_u = g_{11}\dot{u}, \quad p_v = g_{22}\dot{v} \tag{29}$$

and the Hamiltonian is of the form

$$\mathcal{H} = \frac{1}{2}\left(\frac{p_u^2}{g_{11}} + \frac{p_v^2}{g_{22}}\right) + V(u). \tag{30}$$

From (25) and (29) we obtain

$$\Phi = b(v) p_v \tag{31}$$

and (30) reduces to the effective one-dimensional Hamiltonian

$$\mathcal{H}_{\text{eff}} = \frac{1}{2}\left(\frac{p_u^2}{g_{11}(u)} + \frac{\Phi^2}{a^2(u)}\right) + V(u). \tag{32}$$

Thus, the orbits $u = c$ correspond to the equilibria of $\mathcal{H}_{\text{eff}}$. These equilibria are obtained by

$$p_u = 0, \qquad V_u - \frac{a'(u)}{a^3(u)}\Phi^2 = 0. \tag{33}$$

Eq. (33)b correlates the value of the parameter c along the family of orbits to the corresponding constant value of the integral Φ.

Proceeding further, we shall examine the stability type of the equilibrium points. We consider the system of variational equations

$$\dot{\xi} = \Omega \mathcal{S} \xi \tag{34}$$

where the Ω and $\mathcal{S}$ are the 2×2 matrices

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} \dfrac{\partial^2 \mathcal{H}_{\text{eff}}}{\partial u^2} & \dfrac{\partial^2 \mathcal{H}_{\text{eff}}}{\partial p_u \partial u} \\ \dfrac{\partial^2 \mathcal{H}_{\text{eff}}}{\partial u \partial p_u} & \dfrac{\partial^2 \mathcal{H}_{\text{eff}}}{\partial p_u^2} \end{pmatrix} \tag{35}$$

respectively. The values of the second order derivatives of $\mathcal{H}_e$ are calculated at the equilibrium points. So, we have to find the eigenvalues μ of the matrix

$$\mathcal{A} = \Omega \mathcal{S} = \begin{pmatrix} 0 & \dfrac{1}{g_{11}} \\ -\dfrac{\partial^2 \mathcal{H}_{\text{eff}}}{\partial u^2}\bigg|_{p_u=0} & 0 \end{pmatrix}. \tag{36}$$

Its characteristic equation is

$$\mu^2 + \frac{1}{g_{11}} \frac{\partial^2 \mathcal{H}_{\text{eff}}}{\partial u^2}\bigg|_{p_u=0} = 0. \tag{37}$$

By taking into account (32) and eliminating Φ from (33), the condition for linear stability (i.e. imaginary μ) is

$$V_u \left\{ \frac{3a_u}{a} - \frac{a_{uu}}{a_u} \right\} + V_{uu} > 0. \tag{38}$$

On the other hand, by taking into account the inequality $E - V \geq 0$, from Eq. (6) we obtain

$$\frac{g_{22} V_u}{W} \geq 0, \qquad W = \frac{1}{2}(g_{22})_u.$$

So, the motion is allowed in the domain where inequality

$$\frac{V_u}{(\log g_{22})_u} \geq 0 \tag{39}$$

is valid. So we conclude the results of this section with the following:

Proposition 1. *Consider the two-dimensional surface S, submersed in $\mathbb{E}^3$ and let u, v be (curvilinear in general) coordinates on S. Assume that the lines $v = const.$ are geodesics while $u = const.$ are their orthogonal trajectories. Assume also that the coordinate system u, v on S is isothermic. If the family of geodesics is a permissible family of orbits for a material point, moving on S under the influence of the potential V, then:* (a) *the potential is of the form $V = V(u)$ and is integrable with an integral linear in the velocities and* (b) *the orthogonal trajectories $u = const.$ are also orbits compatible with V. They are traced on S in the region defined by inequality (39) and they are linearly stable if inequality (38) is satisfied.*

Remark 1. We note here that the integral of motion (14) arises from the variational symmetry

$$\vec{w} = b(v) \frac{\partial}{\partial v}$$

of the Lagrangian (28) through Noether's theorem (e.g. [15], p. 277).

The first prolongation of $\vec{w}$ is:

$$pr^{(1)}(\vec{w}) = \vec{w} + \frac{\mathrm{d}b}{\mathrm{d}v}\dot{v}\frac{\partial}{\partial \dot{v}}$$

and acting on $\mathcal{L}$, it gives

$$pr^{(1)}(\vec{w})(\mathcal{L}) = 0.$$

4. Existence of integrals of motion quadratic in the velocities

We shall seek for integrals quadratic in the velocity components, i.e. integrals of the form

$$\Phi(u, v, \dot{u}, \dot{v}) = A(u, v)\dot{u}^2 + 2B(u, v)\dot{u}\dot{v} + C(u, v)\dot{v}^2 + D(u, v). \tag{40}$$

Again, due to the definite parity of the Lagrangian with respect to time inversion, the integral cannot possess terms linear in the velocities. In this case, we obtain the following relation

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= A_u\dot{u}^3 + (A_v + 2B_u)\dot{u}^2\dot{v} + (C_u + 2B_v)\dot{u}\dot{v}^2 + C_v\dot{v}^3 \\ &\quad + 2A\dot{u}\ddot{u} + 2C\dot{v}\ddot{v} + 2B\dot{u}\ddot{v} + 2B\dot{v}\ddot{u} + D_u\dot{u} + D_v\dot{v} \equiv 0.\end{aligned} \tag{41}$$

Now, we replace the expressions of $\ddot{u}$, $\ddot{v}$ from (13) into (41) and obtain

$$K_{30}\dot{u}^3 + K_{21}\dot{u}^2\dot{v} + K_{12}\dot{u}\dot{v}^2 + K_{03}\dot{v}^3 + K_{10}\dot{u} + K_{01}\dot{v} \equiv 0 \tag{42}$$

where

$$\begin{aligned}
K_{30} &= A_u - A\frac{\partial(\log g_{11})}{\partial u},\\
K_{21} &= A_v + 2B_u - 2B\frac{\partial(\log g_{22})}{\partial u} - B\frac{\partial(\log g_{11})}{\partial u},\\
K_{12} &= C_u + 2B_v + A\frac{(g_{22})_u}{g_{11}} - 2C\frac{\partial(\log g_{22})}{\partial u} - B\frac{\partial(\log g_{22})}{\partial v},\\
K_{03} &= C_v - C\frac{\partial(\log g_{22})}{\partial v} + B\frac{(g_{22})_u}{g_{11}},\\
K_{10} &= -\frac{2A}{g_{11}}V_u - \frac{2B}{g_{22}}V_v + D_u,\\
K_{01} &= -\frac{2B}{g_{11}}V_u - \frac{2C}{g_{22}}V_v + D_v.
\end{aligned}$$

Since (42) holds by identity, we must have

$$K_{30} = K_{21} = K_{12} = K_{03} = K_{10} = K_{01} = 0. \tag{43}$$

From equation $K_{30} = 0$, we find for the unknown function $A(u, v)$

$$A(u, v) = m(v)g_{11} \tag{44}$$

where $m(v)$ is an arbitrary function to be determined. In the following we shall try to find the other arbitrary functions $B(u, v)$, $C(u, v)$ and $D(u, v)$.

At this point we assume that the geodesics $v = c$ are not anymore orbits allowed by the potential, but now it is compatible with the family of the orthogonal trajectories $f(u, v) = u = c$.

Then we have $f_u = 1$, $f_v = 0$ and taking into account (10) and (11), the quantities α and β in (8) are written as:

$$\alpha = \frac{g_{22}}{(g_{22})_u}, \quad \beta = 0 \tag{45}$$

and Eq. (8) becomes

$$\alpha V_{uv} + \alpha_v V_u + V_v = 0. \tag{46}$$

From the relations $K_{10} = 0$ and $K_{01} = 0$, we take

$$\begin{aligned} \frac{A}{g_{11}} V_u + \frac{B}{g_{22}} V_v &= \frac{D_u}{2}, \\ \frac{B}{g_{11}} V_u + \frac{C}{g_{22}} V_v &= \frac{D_v}{2}. \end{aligned} \tag{47}$$

The compatibility condition for system (47) with respect to D leads to the equation

$$\begin{aligned} &\left(-\frac{B}{g_{11}}\right) V_{uu} + \left(\frac{A}{g_{11}} - \frac{C}{g_{22}}\right) V_{uv} + \left(\frac{B}{g_{22}}\right) V_{vv} \\ &\quad + \left[\frac{A_v}{g_{11}} - \left(\frac{B}{g_{11}}\right)_u\right] V_u + \left[\left(\frac{B}{g_{22}}\right)_v - \left(\frac{C}{g_{22}}\right)_u\right] V_v = 0. \end{aligned} \tag{48}$$

Comparing Eq. (48) with Eq. (46), we take

$$B(u, v) = 0, \tag{49}$$

while, from the relation $K_{21} = 0$, we have $A_v = 0$. Hence, we take for the arbitrary function $m(v)$

$$m(v) = c_1' = \text{const}. \tag{50}$$

Here we shall distinguish two cases:

- (i) $c_1' = 0$. Then $A(u, v) = 0$. From the relations $K_{12} = 0$ and $K_{03} = 0$, we obtain for the unknown function C

  $$C(u, v) = n(v) g_{22}^2, \qquad C(u, v) = l(u) g_{22}. \tag{51}$$

  Thus, the component of metric tensor g_{22} must equal

  $$g_{22} = \frac{l(u)}{n(v)}. \tag{52}$$

  For reasons of consistency with the notation of the previous section, we put

  $$l(u) = a^2(u), \quad n(v) = b^2(v). \tag{53}$$

  From (52) we see that also in this case the coordinate system u, v must be isothermic. Combining relations (51)–(53), we determine function $C(u, v)$,

  $$C(u, v) = \frac{a^4(u)}{b^2(v)}. \tag{54}$$

  Now, we come back in (47) to determine the function D. From Eq. (47)a we find that

  $$D(u, v) = D(v) \tag{55}$$

and from (47)b we determine the potential $V = V(u, v)$ as

$$V(u, v) = \frac{F(u) + D(v)}{2a^2(u)} \tag{56}$$

where now $F(u)$, $D(v)$ are arbitrary functions. The integral of motion is given by

$$\Phi(u, v, \dot{u}, \dot{v}) = \frac{a^4(u)}{b^2(v)}\dot{v}^2 + D(v). \tag{57}$$

- (ii) $c_1' \neq 0$. Without loss of generality, we set $c_1' = 1$ and we have $A(u, v) = g_{11}(u)$. Then we construct the new integral of motion

$$\Phi' = \Phi - 2E \tag{58}$$

where E is the energy integral. In this new form of the integral, $A = 0$ so this case reduces to the previous one.

For the allowed region we observe the inequality $E - V > 0$ and Eq. (6) yields again the condition (39). In order to determine the stability of the orbits, we construct the effective Hamiltonian, which in this case is

$$\mathcal{H}_{\text{eff}} = \frac{1}{2}\left(\frac{p_u^2}{g_{11}} + \frac{\Phi}{a^2(u)}\right) + G(u) \tag{59}$$

where $G(u) = F(u)/(2a^2)$. Working exactly as in the previous section, we find the condition for linear stability

$$G_u\left\{\frac{3a_u}{a} - \frac{a_{uu}}{a_u}\right\} + G_{uu} > 0. \tag{60}$$

We conclude this section with the following:

Proposition 2. *Under the assumptions of Proposition 1, if the family of trajectories, orthogonal to the family of geodesics, is a permissible family of orbits for a material point, moving under the influence of the potential V, then the potential is of the separable form (56) and is integrable with an integral quadratic in the velocities, of the form (57). The orbits are traced in the region defined by inequality (59) and they are linearly stable if inequality (60) is satisfied.*

Remark 2. We note here that if $a(u) = \text{const.}$, then the Hamiltonian (59) can be written as a sum of two integrals of motion. In this case, we have

$$\mathcal{H}_{\text{eff}} = S(u, p_u) + T(u, p_v)$$

where

$$S(u, p_u) = \frac{p_u^2}{2g_{11}(u)} + \frac{F(u)}{2a^2}, \quad T(v, p_v) = \frac{\Phi}{2a^2}$$

and $S(u, p_u)$ is also an integral of motion.

Remark 3. The quadratic integral (57) arises from the generalized symmetry (e.g. [15], pp. 325–330)

$$\vec{w}_Q = \frac{2a^4(u)}{b^2(v)}\dot{v}\frac{\partial}{\partial v}$$

such that

$$pr^{(1)}(\vec{w}_Q)(\mathcal{L}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2a^4(u)}{b^2(v)}\dot{v}\frac{\partial \mathcal{L}}{\partial \dot{v}} - \frac{a^4(u)}{b^2(v)}\dot{v}^2 - D(v)\right).$$

5. Examples

In this section we shall give some pertinent examples:

Example 1. We consider motion on $\mathbb{R}^2$ and assign the metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

where r and θ are polar coordinates. We consider the family of straight lines $\theta = c$, which are geodesics, and the orthogonal family of circles $r = c$. Let us consider that the lines $\theta = c$ are permissible orbits. Then the potential is central, i.e. $V = V(r)$ and there exists an integral of motion linear in the velocities, $\Phi = r^2\dot{\theta}$, i.e. the angular momentum integral. The circles $r = c$ are also permissible orbits and Eqs. (38) and (39) for the allowed domain and their stability reduce to the well known conditions $V_r > 0$ and $V_{rr}/V_r + 3/r > 0$. If, on the other hand, we consider that only the circles $r = c$ are permissible orbits, then the potential is of the form

$$V = \frac{F(r) + D(\theta)}{2r^2}$$

with $F(r)$, $D(\theta)$ arbitrary functions, which is separable in polar coordinates, and there exists an integral, quadratic in the velocities,

$$\Phi = r^4\dot{\theta}^2 + 2D(\theta).$$

These results are already known as special cases in Ichtiaroglou and Meletlidou [10]. The compatibility of the above potential with the family of circles was initially given in Broucke and Lass [6].

Example 2. We consider motion on the two-dimensional unit sphere S^2 with spherical coordinates θ, ϕ. The line element is $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. The meridians $\phi = c$ are geodesics and the parallels $\theta = c$ are orthogonal to them. If we assume that the meridians are permissible orbits for a point mass, the potential is of the form $V = V(\theta)$ and there exists the integral of motion $\Phi = \sin^2\theta\,\dot{\phi}$, linear in the velocities. If on the other hand, motion on the parallels is permissible, the potential is of the form

$$V = \frac{h(\theta) + g(\phi)}{2\sin^2\theta}$$

and the integral of motion is

$$\Phi = \sin^4\theta\dot{\phi}^2 + 2g(\phi).$$

These results have been recently obtained by Voyatzi and Ichtiaroglou [17].

Example 3. We consider motion on the cone defined by $\vec{r}\,(u, v) = \{u\cos v, u\sin v, u\}$. The line element is

$$\mathrm{d}s^2 = 2\mathrm{d}u^2 + u^2\mathrm{d}v^2.$$

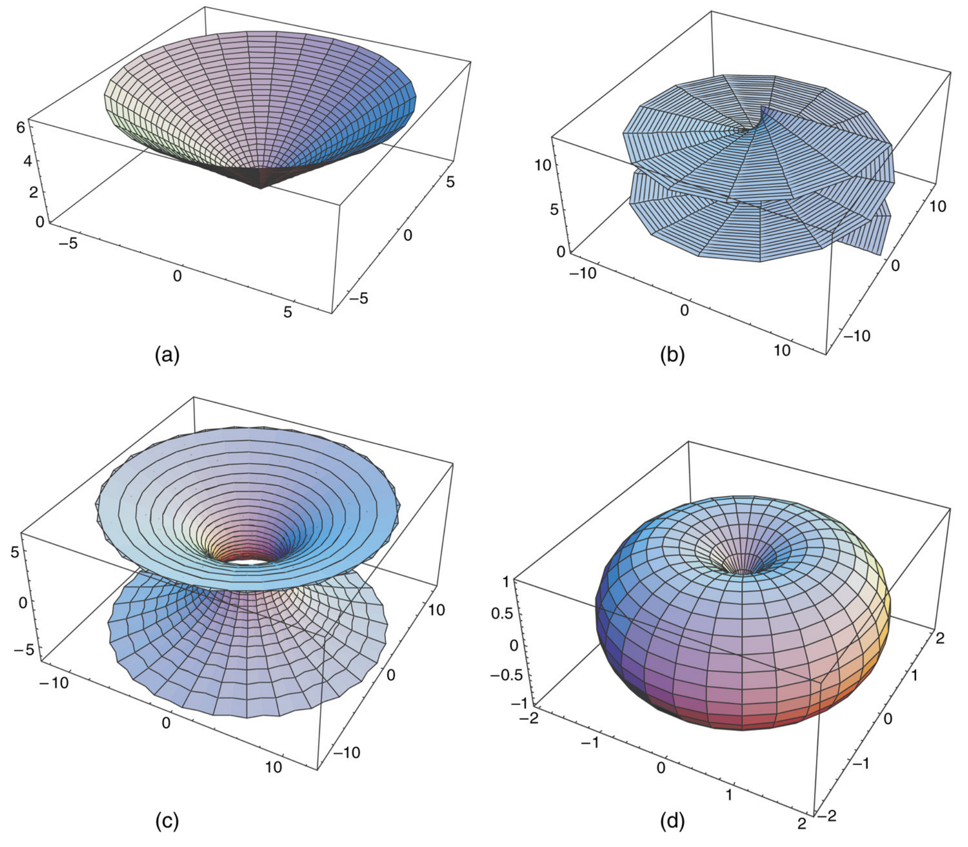

Fig. 1. Surfaces in the Examples 1–6. (a) Cone, (b) Helicoid surface, (c) Catenary surface and (d) Torus with $a = b = 1$.

The straight lines $v = c$ are geodesics and the circles $u = c$ are orthogonal to them, see Fig. 1(a). If the geodesics are orbits, compatible with the potential, then it is of the form $V = V(u)$ and there exists an integral of motion linear in the velocities, $\Phi = u^2 \dot{v}$. If, on the other hand, motion on the circles is allowed, the potential must have the form

$$V = \frac{F(u) + D(v)}{2u^2}$$

and the second integral of motion is

$$\Phi = u^4 \dot{v}^2 + D(v).$$

Example 4. For the helicoid surface defined by $\vec{r}(u, v) = \{u \cos v, u \sin v, v\}$, the line element is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + (1 + u^2)\mathrm{d}v^2.$$

The geodesics $v = c$ are straight lines while the helical lines $u = c$ are orthogonal to them, see Fig. 1(b). Motion on the straight lines leads to the potential $V = V(u)$ and the integral $\Phi = (1 + u^2)\dot{v}$ while motion on the helical lines is compatible with the potential

$$V = \frac{F(u) + D(v)}{2(1 + u^2)}$$

and the integral

$$\Phi = (1+u^2)^2\dot{v}^2 + D(v).$$

Example 5. The line element on the catenary surface

$$\vec{r}(u,v) = \left\{3\cosh\left(\frac{u}{3}\right)\cos v, 3\cosh\left(\frac{u}{3}\right)\sin v, u\right\}$$

is given by

$$\mathrm{d}s^2 = \cosh^2\left(\frac{u}{3}\right)\mathrm{d}u^2 + 9\cosh^2\left(\frac{u}{3}\right)\mathrm{d}v^2.$$

The geodesics are the intersections with the meridian planes $v = c$ and the circles $u = c$ are orthogonal to them, see Fig. 1(c). If the geodesics are permissible orbits, the potential is of the form $V = V(u)$ and there exists an integral of motion linear in the velocities,

$$\Phi = 9\cosh^2\left(\frac{u}{3}\right)\dot{v}$$

and if the circles are permissible orbits, then the potential is

$$V = \frac{F(u)+D(v)}{18\cosh^2(\frac{u}{3})}$$

and there exists an integral of motion quadratic in the velocities,

$$\Phi = 81\cosh^4\left(\frac{u}{3}\right)\dot{v}^2 + D(v).$$

Example 6. For motion on the torus

$$\vec{r}(u,v) = \{\cos v[a+b\cos u], \sin v[a+b\cos u], b\sin u\},$$

with $a, b > 0, u \geq 0, v < 2\pi$ we have

$$\mathrm{d}s^2 = b^2\mathrm{d}u^2 + (a+b\cos u)^2\mathrm{d}v^2.$$

The meridian circles $v = c$ are geodesic lines and the horizontal circles $u = c$ are their orthogonal trajectories, see Fig. 1(d). For permissible motion on the geodesics, the potential must be $V = V(u)$ and there exists an integral of motion linear in the velocities $\Phi = (a+b\cos u)^2\dot{v}$, while for motion on the horizontal circles, the potential must be of the form

$$V = \frac{F(u)+D(v)}{2(a+b\cos u)^2},$$

with the integral

$$\Phi = (a+b\cos u)^4\dot{v}^2 + D(v).$$

6. Conclusions

In this paper we study the motion of a material point on a two-dimensional surface. We select orthogonal coordinates u, v, such that the lines $v =$ const. are geodesics, while $u =$ const. are their orthogonal trajectories and we assume that the coordinate system is isothermic. We prove that, if the family of geodesics is a family of orbits of the material point, compatible with the potential, then it is integrable with an integral linear in the velocities, while compatibility of the potential with the orthogonal trajectories guarantees integrability with a quadratic integral of motion. In both cases, we determine the form of the potential and the corresponding form of the integral, while, for the case of the orthogonal trajectories, we determine the allowed regions of motion on the surface and the stability of the orbits. We also show that in the first case where the geodesics are permissible orbits, their orthogonal trajectories are also compatible with the potential. This is consistent with Proposition 2, since the systems of Proposition 1 are also integrable with a "quadratic" integral of motion, if one considers the square of the linear integral. Note that the potentials of Propositions 1 and 2 are of a form, separable in the coordinates u, v.

The compatibility of these orbits with the potential is only a sufficient condition for separability. From the case of planar motion, where the existence of suitable families of conics guarantees also separability [10] we know that this condition is in general not also necessary. It would be interesting to search for the existence of other permissible monoparametric families of orbits with this property, for the case of motion on a two-dimensional surface in a general setting.

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Appendix

Here we supply the necessary formulas for the various functions in Section 2, for the general case of motion of a material point on an arbitrary surface [3].

$$\alpha = \frac{1}{2W}(g_{22} f_u - g_{12} f_v), \quad \beta = \frac{1}{2W}(-g_{12} f_u + g_{11} f_v), \quad \gamma = \frac{f_v}{f_u},$$

$$W = \frac{1}{A}[g(f_v^2 f_{uu} - 2 f_u f_v f_{uv} + f_u^2 f_{vv}) - B_1(g_{22} f_u - g_{12} f_v) - B_2(g_{11} f_v - g_{12} f_u)],$$

$$A = g_{11} f_v^2 - 2 g_{12} f_u f_v + g_{22} f_u^2,$$

$$B_1 = \frac{1}{2}(g_{11})_u f_v^2 + \left[(g_{12})_v - \frac{1}{2}(g_{22})_u\right] f_u^2 - (g_{11})_v f_u f_v,$$

$$B_2 = \left[(g_{12})_u - \frac{1}{2}(g_{11})_v\right] f_v^2 + \frac{1}{2}(g_{22})_v f_u^2 - (g_{22})_u f_u f_v,$$

$$g = g_{11} g_{22} - (g_{12})^2.$$

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